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Research article

An Analytical and Coherent approach of Hybrid Analysis of Heat Conduction

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Abstract

This paper deals with current *techniques* of solution by means of hybrid analyses that combine analytical and numerical approaches. This innovative procedure, however, requires a full revision of existing engineering methods which ultimately, will result in the hybrid method of the future. In these studies, the generalized integral transform technique, which is typical method, is used to solve heat convection problems with Mathematica. In other words, in Mathematica files, where all steps of the solution are carried out. In fact, notebook can be read as plain text but also be utilized to execute the input statements and to produce tables of results, graphic and animations. In turn, by changing the input statements, the calculations can be repeated to obtain new results. On the other hand, the examples of applications deals with the classical Graetz problems of stationary forced convection in thermally developing hydrodynamically developed laminar follows in a parallel plate channel and in a circular tube.

Key words: Hybrid analysis, generalized integral transform technique, Nusselt number, parallel plate channel, Mathematica, hybrid methods in engineering.

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1 Introduction

In this paper we studied in numerical computation which is completely dominated the science and engineering. But over the past decade powerful computer systems have been widely available to carry out hybrid computation.

[1, 2]. This software opens a new avenue for solving complicated engineering problems using hybrid analysis that combine analytical and numerical approaches. A full revision of the existing engineering methods is needed, such software may provide to dominate the hybrid methods in the future. Now, this text is created by using original notebooks documents [3], where all steps of the solutions- statements of the problem, nondimensionalizing, eigenproblem, integral transform pair, transforming to ordinary differential equations, solutions, computational rules, and results are made by using *Mathematica* [1]. Furthermore, notebooks can be used not to read but also to execute the input statements, plot graphics and animation. The goal of the present paper is to demonstrate that the generalized integral transform technique (GITT), which is a typical hybrid method could be efficiently used to solve heat convection problems with *Mathematica* and for the latest development in the GITT [4,7].

Hence considering the classical Graetz problems of steady state heat transfer in thermally developing , hydrodynamically developed forced laminar flow inside a parallel plate channel. The exact solution in terms of the Kummer confluent hypergeometric function is presented.

2 Exact Solution for Parallel Plate Channel Flow

We consider the classical problem of steady state heat transfer in thermally developing, hydrodynamically developed forced laminar flow inside a parallel plate channel under the following

(i) The fluid is incompressible
(ii) The velocity profile is fully developed and known
(iii) The entrance temperature is uniform
(iv) The temperature of the channel wall is a constant

Now the dimensional temperature $\theta[\xi,\eta]$ of a fluid along the channel $0\leq \eta \leq \infty$ in the region $0\leq \xi \leq 1$, [8]

$$w[\xi]\ \frac{\partial\theta[\xi,\eta]}{\partial\eta} = \frac{\partial^2}{\partial\xi^2}[\xi,\eta] \tag{1}$$

$$\frac{\partial\theta\ [0,\eta]}{\partial\xi} = 0 \tag{2}$$

$$\theta[1,\eta] = 0 \tag{3}$$

$$\theta[\xi,0] = 1 \tag{4}$$

The problems from Eqs. (1)after pioneering works [9-11] is referred to as a Graetz Nusselt type of problem. The velocity profile for laminar flow is [8]

$$w\ [\xi] = \frac{3}{2}\ (1-\xi^2) \tag{5}$$

Once $\theta[\xi,\eta]$ is determined, the average temperature $\theta av[\ \eta]$ and Nusselt number $Nu[\ \eta]$ are evaluated from [8],

$$\theta av[\ \eta] = \frac{\int_0^1 w\ [\xi]\ \theta[\xi,\eta]\ d\xi}{\int_0^1 w\ [\xi]\ d\xi} \tag{6}$$

$$Nu\,[\eta] = -\frac{\frac{\partial\theta[1,\eta]}{\partial\xi}}{\theta av[\eta]} \tag{7}$$

To solve the problem Eq.(1) we apply the following eigenvalue problem,

$$\psi_1[\xi] + \mu_i^2\, w[\xi]\, \psi_i[\xi] = 0 \tag{8}$$

$$\psi_i'\,[0] = 0 \tag{9}$$

$$\psi_1[1] == 0 \tag{10}$$

again with the normalization condition

$$\int_0^1 w[\xi]\, \psi_1[\xi^2]\, d\xi = 1 \tag{11}$$

The solution of the $\theta[\xi,\eta]$ has the form

$$\theta[\xi,\eta] = \sum_{i=1}^{n} \theta_i[\eta]\, \psi_i[\xi] \tag{12}$$

Where the summation is taken to the truncation number n and

$$\theta_i[\,\eta] = \int_0^1 w[\xi]\, \theta\,[\xi,\eta]\, \psi_1[\xi]\, d\xi \tag{13}$$

Eq.(13) is the finite integral transform of the function $\theta[\xi,\eta]$ with respect to the space variable ξ and Eq.(12) is called inversion formula of the finite integral transform Eq. (13). To transform the partial differential equation problem Eq. (1) into an ordinary differential equation for the integral transform we use the identity

$$\int_0^1 \psi_i[\xi]\, \frac{\partial^2\theta}{\partial\xi^2}[\xi,\eta] d\xi - \int_0^1 \theta\,[\xi,\eta]\, \psi_1[\xi] d\xi$$

$$= \theta[0,\eta]\psi_i'\,[0] - \theta[1,\eta]\psi_i'\,[1] - \psi_i\,[0]\frac{\partial\theta}{\partial\xi}\,[0,\eta] + \psi_i[1]\frac{\partial\theta}{\partial\xi}[1,\eta] \tag{14}$$

From Eqs. (1)–(3) we find $\frac{\partial^2\theta}{\partial\xi^2}[\xi,\eta], \frac{\partial\theta}{\partial\xi}[0,\eta]\ and\ \theta[1,\eta]$. From Eq.(10) we find $\psi_i[\xi], \psi_i'\,[0]\ and\ \psi_i\,[1]$, the results are introduced into Eq. (14) we obtain

$$\theta_i'[\eta] + \mu_i^2\, \theta_i\,[\eta] = 0 \tag{15}$$

The entrance condition Eq. (4) is transformed to

$$\theta_i = -\frac{\psi_i'}{\mu_i^2}\,[1] \tag{16}$$

The solution of equations (15)-(16), replaced into the inversion formula, Eq.(12) gives the temperature

$$\theta[\xi,\eta] = -\sum_i^n \frac{\psi_i'}{\mu_i^2}[1]\psi_i[\xi]\, e^{-\eta\mu_i^2} \tag{17}$$

To obtain the average temperature we replace Eqs.(17)into Eq.(6) then

$$\theta av\,[\eta] = \sum_{i=1}^n \frac{\psi_i'}{\mu_i^4}[1]^2\, e^{-\eta\mu_i^2} \tag{18}$$

To obtain the Nusselt number we replace Eqs. (17) and (18) into Eq. (9),

$$Nu[\eta] = \sum_{i=1}^n \frac{\psi_i'}{\mu_i^2} \frac{[1]^2 e^{-\eta\mu_i^2}}{\sum_i^n \frac{\psi_i'}{\mu_i^4} e^{-\eta\mu_i^2}} \tag{19}$$

The limiting Nusselt number is

$$Nu[\infty] = \mu_1^2 \tag{20}$$

Introducing $\upsilon_i^2 = \frac{3}{2}\mu_i^2$ and equation (5) into equation (8) we obtain

$$\psi_i\,[\xi] + \upsilon_i^2(1\text{-}\xi^2)\,\psi_i\,[\xi] = 0 \tag{21}$$

The exact solution of Eq. (21) into Eq. (9) is

$$\psi_i\,[\xi] = \text{C Hypergeometric 1F1}\left[\frac{1-\upsilon[i]}{4},\ \frac{1}{2},\ \upsilon[i]\xi^2\right]\text{Exp}\left[\frac{\upsilon[i]\xi^2}{2}\right] \tag{22}$$

Where Hypergeometric 1F1 [a, b, z] $\left[\frac{1}{4}\ (1-\upsilon_i\ ), \frac{1}{2},\ \ \upsilon_i\right] = 0$ (23)

The normalization equation (11) gives

$$C[i]^2 = \frac{2}{3\int_0^1 e^{-\xi^2}\upsilon_i\ (1-\xi^2)\ Hypergeometric\ 1F1\left[\frac{1}{4}(1-\upsilon_i), \frac{1}{2,}\ \xi^2\ \upsilon_i\right]^2 d\xi} \quad (24)$$

The solution obtained in the above is programmed and plotted in the forthcoming section.

3 Rules and Computations related to Section 2

Moreover, the *Mathematica* rules for the exact solution has been obtained and these rules can be enterered in *Mathematica* 3 or *Marthematica* 4 to repeat our calculation and obtain new results. Now the temperature given by Eq. (17)

is programmed as

$\theta[-,0|\ 0|,-] := 1$

$\theta\ [[1|1|,-,-] := 0$

$\theta\ [\ \xi - ?\,Numeric\ Q, \eta - ?\,NumericQ, n\ Integer\ ?\,Positive\ ] :=$

$\sum_{i=1}^{n} a[i]\ \psi\ [i]\ [\xi]\ Exp\ [-\mu\ ][i]^2\ \eta$

Where $a\ [i\ -] := a[i]\ =\ -\ \frac{\psi[i]\prime}{\mu[i]^2}[1]$

$\mu[i\ -] :=\ \sqrt{\frac{2}{3}}\ \ \upsilon[i]$

Now the eigen values are computed once and stored in the memory by the rule,

$\upsilon\ [i\ -]: = \upsilon\ [i]$

$\frac{x}{.}FindRoot\ \left[Exp\ [-\frac{x}{2}\right]\ Hypergeometric1F1\ \left[\frac{1-x}{4}, \frac{1}{2}, x\right] == 0,$

{x, 4 {i-1, i},

AccuracyGoal →5]

Now the eigenfucntions ar computed by

$\psi\ [i\ -][\xi\ -]: =$

C1[i] Hypergeometric1F1$\left[\frac{1-\upsilon(i)}{4},\ \ \frac{1}{2},\ \upsilon[i]\xi^2\right]\ Exp\ \left[-\frac{\upsilon[i]}{2}\ \xi^2\right]$

Where

C1[i-] := C1 [i] =

Sqrt [2/(3 NIntegrate [Evaluate [$e^{-\upsilon(i)}\xi^{\wedge 2}\ \left(1-\xi^{\wedge 2}\right) Hypergeometric\ 1F1\ [(1-\upsilon[i])/4,$

1/2, $\upsilon[i]\xi^{\wedge 2}]^{\wedge}2],\ \{\xi, 0, 1\}])]$

The first 12 roots coincide with these reported by Brown [12] and repreinted in [8]

Table $\upsilon\ [i], \{1, 1, 12\}]$

{1.6816, 5.66986, 9.66824, 13.6677, 17.6674, 21.6672, 25.6671, 29.667, 33.667, 37.6669, 41.6669, 45.6669}

The first 12 normalization *constants* are

Table [C1[i], {I, 1, 12}]

{1.25857, 1.29773, 1.3011, 1.30202, 1.30239, 1.30257, 1.30268, 1.30275, 1.30279, 1.30282, 1.30284, 1.30286}

The dimensionless temperature at ξ from 0 to 1 step 0.1 and η= 0.2, calculated by using 12 tems, are

Table [θ [ξ, 0.2, 12], { [ξ, 0, 1, 0.1 }]

{0.819525, 0.80857, 0.776045, 0.723012, 0.651309, 0.563547, 0.462973, 0.353162, 0.237598, 0.119216, 0}

For smaller values of η more terms are needed the plot of the temperature field is

Plot 3D[θ [ξ,η, 12], {η, 0.01, 1.5},

AxesLabel→{η, ξ, θ}, Shading→ False, Boxed →False];

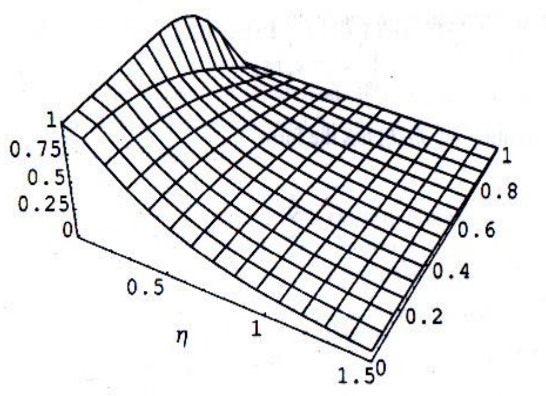

Figure 1. Temperaure distribution $\theta[\xi,\eta]$ inside a parallel plate channel

The average temperaturefrom Eq. (18) is programmed as

θav [0 | 0., -] : = 1

θav [η- ? NumericQ, n-Integer? Positive] := $\sum_{1=1}^{n} a\,[i]^2\ Exp\ [-\mu[i]^2\,\eta\,]$

The average temperature at η from 0 to 1 step 0.05 calculated by using 12 terms are

Table [θav [η, 12], { η, 0, 1, 0.05 }]

{1, 0.847354, 0.760204, 0.688258, 0.625134, 0.568487, 0.517208, 0.470635, 0.428283, 0.389752, 0.354691, 0.322785, 0.293749, 0.267326, 0.243279, 0.221395, 0.20148, 0.183357, 0.166863, 0.151853, 0.138194}

The plot of the average temperature is

Plot [θav [η, 12],{η , 0, 2}, PlotRange → {0, 1} , AxesLabel → { η, θav}];

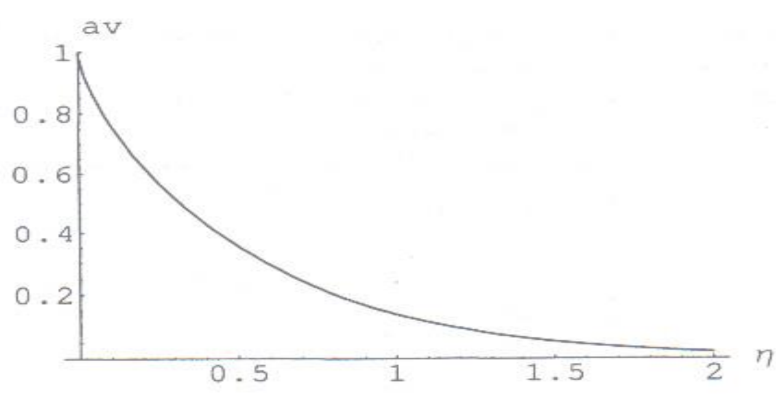

Fig.(2) Average temperature θav [η] for a parallel plate channel

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The local Nusselt numbers at η from 0 to 0.7 step are

Table [Nu [η, 12], { η, 0, 0.7, 0.05}]

{∞, 2.35529, 2.047783, 1.94591, 1.90804, 1.89378, 1.88842, 1.88639, 1.88563, 1.88535, 1.88524, 1.8852, 1.88518, 1.88518}

The plot of the Nusselt number is

Plot[Nu[η, 12], {η, 0.001, 0.1 } , AxesLabel→ (η, Nu})] ;

Fig(3). Nusselt number Nu[η] of a parallel plate channel

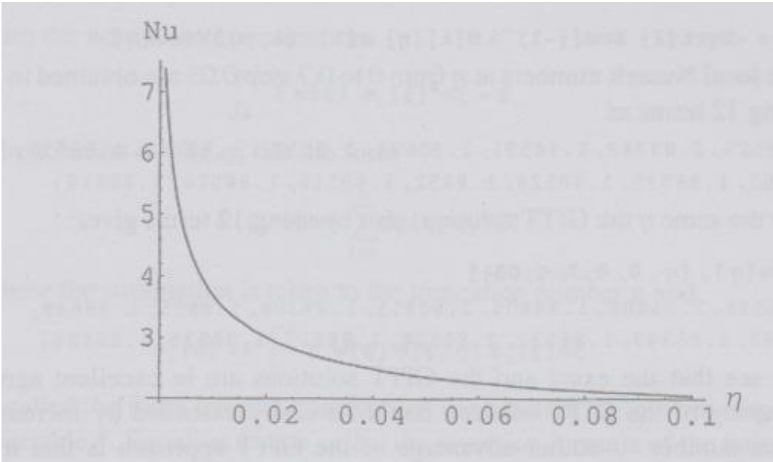

Figure 3. Nusselt number Nu[η] of a parallel plate channel

4 GITT Solutions for Parallel Plate Channel Flow

Generalized integral transform technique is typical hybrid method and in our present studies
is similar to the method proposed by Siegel [13] from Eq.(1) by GITT, we use the normalized eigenfuntions by Eq.(8) for the case w[ξ] =1

$$\psi_i[\xi] = \sqrt{2}\ \mathrm{Cos}\ [\mu_1 \xi\ ] \qquad (25)$$

Where the eigen values are as

$$\psi_i = \left(i - \tfrac{1}{2}\right)\pi \qquad (26)$$

$$\int_0^1 w\ [\xi]\ \psi_i[\xi]\ \frac{\partial\theta}{\partial\eta}[\xi,\eta] d\xi + \mu_i^2\ \theta_i\ [\eta] == 0 \qquad (27)$$

The temperature gradient in the above integral according to GITT, is is removed by using the inversion formula by Eq. (12), we obtain

$$\sum_{j=1}^{n} W\ [\ i,j]\theta_j'\ [\ \eta] + \ \mu_i^2\ {}_{\theta_i}[\eta] == 0 \qquad (28)$$

Where

$$\mathrm{W\ [i,\ j]} = \int_0^1 w[\xi]\psi_{\ i}\ [\xi]\ \psi_j\ [\xi]\ d\xi \qquad (29)$$

The entrance condition Eq. (4) is transformed to the following equation

$$\theta_{\ i\ [0]} == -\frac{(-1)^i}{\mu_i}\sqrt{2} \qquad (30)$$

Introducing Eqs. (5) and (25) into Eq. (29) after solving the integral one obtains :

$$W[i, j] = 1 + \frac{3}{4\,\mu_i^2}, \quad W[i,j] \longrightarrow -\frac{12\,(-1)^{i+j}}{\left(\mu_i^2 - \mu_j^2\right)^2}\mu_i\,\mu_j \tag{31}$$

Eq.(27) can be rewritten in matrix form

$$[w]\,\{\theta\}' + [M]\{\theta\} = 0 \tag{32}$$

Once the integral transform θ_i [η] are found, the desired temperature θ [ξ, η] is computed by the inversion equation (12) then the average temperature θ av [η] and Nusselt number Nu [η] are determined by using Eqs. (6)-(7).

Conclusion and Discussion

In the present paper we have demonstrated the power of the generalized integral transform technique which is a typical hybrid method. The present texts adapted to the classical paper format applying original notebooks documents where all steps, of the solutions, statement of the problems, nondimensionalizing, eigenproblems, integral transform pair, transforming to ordinary differential equation, solution, computational rules and results are made using Mathematica. Moreover, hybrid computation, combining the power of analytical and numerical approaches, is a very promising and thrust area research and applications, (ww.begellhouse.com)

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